LEARNING CONCEPTS THROUGH THE HISTORY OF MATHEMATICS

The Case of Symbolic Algebra

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Abstract: The adolescent’s notion of rationality often encompasses the epistemological view of mathematics as knowledge which offers absolute certainty. Several findings such as Gödel’s theorem and the construction of a strict finite arithmetic, however, provide strong arguments against that view. The static and unalterable mode of presentation of concepts in the mathematics curriculum, rather than lack of knowledge of metatheory, contributes to this misconception. I will argue that the conceptual history of mathematics provides excellent opportunities to convey the basic epistemological and ontological questions of the philosophy of mathematics in mathematics education. In particular, the emergence of the concept of an equation will be presented in a historical context. Such examples will alert students of the relativity of mathematical methods, truth, and knowledge, and will put mathematics back in the perspective of time, culture, and context.

Key words: History of algebra, concept formation, absolute truth, inconsistency

1. INTRODUCTION

In this chapter, we argue for the integration of the history of mathematics in mathematics education. Although history is not the same as philosophy, we believe that the history of mathematics provides many opportunities to convey basic philosophical concepts concerning the epistemological and ontological aspects of mathematics.

On the epistemological level, a conceptual history of mathematics raises questions such as:
How are concepts formed in mathematics?
Which factors influence or change the meaning of concepts?
Is there an internal logic and order in the development of mathematical concepts?
What is the role of symbolism in mathematical knowledge?
What constitutes a valid proof in mathematics?

Concerning ontology, the history of mathematics provides challenging arguments in the realism-constructivism debate. If the meaning of basic mathematical concepts, such as number or equation, changes during the development of mathematics, what happens to the ontological status of these concepts? We will explore how these questions can be approached within the teaching of elementary algebra.

Several studies have been published on the use of the history of mathematics in mathematics education. Arguing for the use of the history of mathematics may therefore seem to be a superfluous task. The official curriculum for secondary education in Flanders (Belgium) defines the role of the history of mathematics explicitly:

Mathematics education is necessarily connected with other disciplines. Mathematics itself has developed through centuries in close connection with prevailing opinions and problems. Today, certain historical contexts still provide useful starting points to approach specific mathematical concepts and educational topics. The historical context shall therefore be integrated in our curriculum.

When looking for concrete guidelines of how to integrate the history of mathematics, however, the examples offered by the plan are disappointing. We only find generalities such as “an approach with examples from architecture and painting can illustrate the role of mathematics in the development of certain art forms” (ibid. 11) and “assignments can be given to research historical facts, such as an internet search for mathematicians, important mathematical theorems, mathematical illustrations and applications” (ibid. 28). The history of mathematics is forced into an illustrative role. History delivers the pictures for lighting up dreary textbooks, to force a connection with other disciplines, and to keep students busy between other assignments. An integrated view on mathematics

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1 Some representative collections are Callinger 1996 and Fauvel and van Maanen 2000.
2 Cited from the education plan of the first two years of secondary school used by Catholic schools (2002, 29, my translation). The education plans of other schools in Belgium and in other European countries employ very similar formulations. For an overview of the place of history in mathematics education in several countries, see the ICMI study of Fauvel and van Maanen 2000, Chapter 1.
education, in which the history of mathematics has a methodological and philosophical relevance, is absent.

We will provide some basic arguments for the integration of the history of mathematics into mathematics education. The first addresses the epistemological status of mathematics. The adolescent’s notion of rationality often encompasses the epistemological view of mathematics as knowledge that offers absolute certainty. He probably has heard of a geometry in which the parallel postulate does not hold, but most likely believes that Euclidian geometry is the real one. We can assume that he is not familiar with Gödel’s theorems and undecidability. It is further unlikely that he has been taught about the existence of inconsistent arithmetic that performs finite calculations as correct as traditional arithmetic. These findings provide strong arguments against the view that mathematics offers absolute truth. The static and unalterable mode of presentation of concepts in the mathematics curriculum, rather than lack of knowledge, contributes to this misconception. Mathematical concepts, even the most elementary ones, have changed completely and repeatedly over time. Major contributions to the development of mathematics have been possible only because of significant revisions and expansions of the scope and contents of the objects of mathematics. Yet, we do not find this reflected in classroom teaching. While the room for integrating philosophy in mathematics education is very limited, an emphasis on the understanding of mathematical concepts is a necessary condition for a philosophical discourse about mathematics. The conceptual history of mathematics provides ample material for such focus, and leads to a better understanding of mathematics and our knowledge of mathematics. I will argue for the integration of the history of mathematics within the mathematics curriculum, as a way to teach students about the evolution and context-dependency of human knowledge. Such a view agrees with the contextual approach to rationality as proposed by Batens (2004). As a prime example, I will treat the development of the concept of a symbolic equation before the seventeenth century. In line with Lakatos (1976) and Kitcher (1984), my example is motivated by the epistemological relevance of the history of mathematics.

2. LIVING WITH INCONSISTENCIES

When asked for an example of an absolute truth, a student might likely answer “one plus one equals two.” This is a grateful example to expand on. One plus one equals two is a current axiomatization of arithmetic, and is therefore true with respect to that theory. It is rather easy, however, to tailor the axiomatization to undermine the truth value of the given statement.
Adapting the Peano axioms leading to one being the successor of one, would yield the example false in the new theory. Given that “one plus one equals two” is true in one theory and not in another refutes the example as an absolute truth. The student might object that changing the rules of arithmetic would lead to complete anarchy in society. The more intelligent student might notice that changing the Peano axioms in the given way would lead to an inconsistent theory, and that anything can be derived from inconsistencies. Let us look at these objections.

The point that changing the truth value of the given example makes no sense might be true, for now. There can be reasons, however, for changing the axioms of arithmetic. Van Bendegem (1994) did develop an inconsistent arithmetic by changing the Peano axioms so that there exists one number that is the successor of itself. His reason for doing so is to demonstrate the feasibility of a strict finite arithmetic. The fifth Peano axiom states that if equality applies for \( x = y \) then \( x \) and \( y \) are the same number. This is the axiom that is tweaked by Van Bendegem so that starting from some number \( n \), all its successors will be equal to \( n \). If we take \( n \) to be one, then in this newly defined arithmetic, \( 1 + 1 = 1 \). That would be a trivial arithmetic, however, which is not the intention of this enterprise. Rather than using one, the number \( n \) can be any number you like. Given a sufficiently large \( n \), all operations of arithmetic behave the same way, as long as this number \( n \) is not reached during calculations. Now, a problem arrives when we reach \( n \). The statement \( n = n + 1 \) is thus both true and false at the same time. This makes the new arithmetic inconsistent.

In classical logic you have the rule ex falso quodlibet (EFQ) which states that \( p \land \neg p \rightarrow q \) or from an inconsistency you can derive anything. This would render the arithmetic trivial within classical logic (CL). Several paraconsistent logics now exist that do not have this problem, as well as inconsistency-adaptive logics developed at the Centre of Logic and Philosophy of Science (Batens 2001). Van Bendegem used the three-valued paraconsistent logic PL from Priest (1987), in which EFQ does not hold. With this underlying logic, he proved that if \( A \) is a valid statement in classical elementary number theory, then \( A \) is also valid in an elementary numbers theory based on a finite model. Gödel proved that every consistent formal theory that is rich enough to model arithmetic will contain true statements that cannot be proved within that theory. In other words, every consistent formal theory is incomplete. Giving up consistency, this new arithmetic, based on a finite model, has the advantage of being complete.

The first axiomatization of arithmetic was given by Giuseppe Peano in a Latin publication of 1889, *Arithmetices principia, nova methodo exposita*. For an annotated English translation, see van Heijenoort 1967, 83–97.
There remains the objection of anarchy. What would happen if some people decided to change the rules of arithmetic? Would our accounting and wage calculation programs become unreliable when working with inconsistent arithmetic? In some sense, we already use this finite and inconsistent arithmetic in computer programs. An unsigned integer in a programming language such as C is represented by a 32- or 64-bit data structure, depending on the underlying hardware. Our inconsistent number \( n \) here becomes \( 2^{32} - 1 \) or \( 2^{64} - 1 \), while its successor is 0. Usually compilers warn for overflow situations such as these. When manipulating the binary structure with bit shift operations, the programmer has to reason within an inconsistent arithmetic and take care of the borderline situations himself. Apparently, many are more worried about giving up absolute certainty in mathematics than they are about their own life by relying on computers in daily situations. We do not have the slightest proof that the current commercial computers and compilers we use to create programs function the way we think they do. Such programs activate the anti-braking system in our car, guide traffic lights, and are used to calculate the structure of bridges and buildings. If they fail to work, human life may be at risk. There are attempts to prove the correctness of hardware design and computer programs, but these are not for practical or commercial use. In fact, we have proof of the contrary. Commercial computers have been known to be inconsistent in their arithmetic, as was shown with the famous Intel Pentium bug. The fact, therefore, is that we live with inconsistencies every day of our life. Why is it so hard to accept this on a philosophical level?

3. **ABSOLUTE CERTAINTY IN MATHEMATICS?**

“Gentleman, that \( e^{i} + 1 = 0 \) is surely true, but it is absolutely paradoxical; we cannot understand it, and we don’t know what it means, but we have proved it, and therefore we know it must be the truth.”

This well-known quote by Benjamin Peirce after proving Euler’s identity in a lecture, reflects the predominant view of mathematicians before 1930, when mathematical truth equalled provability. When Gödel proved that there are true statements in any consistent formal system that cannot be

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4 Given the calculation \( x - \frac{x}{y} \), the first Pentium chip produced the solution \( z = 256 \) for \( x = 4195835 \) and \( y = 3145727 \), instead of the correct \( z = 0 \). For more, see Coe, Tim, et al. 1995.

5 Quoted in Kasner and Newman 1940.
proved within that system, truth became peremptorily decoupled of provability.

Peirce seems to imply something stronger, however, proving things in mathematics leads us to the truth. This goes beyond an epistemological viewpoint and is a metaphysical statement about the existence of mathematical objects and their truth independent of human knowledge. The great mathematician Hardy formulates it more strongly (Hardy 1929):

“It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality.”

Such statements are more than innocent metaphysical reflections open for discussion. They hide implicit values about the way mathematics develops, and have important consequences for the education and research of mathematics. An objective reality implies the fixed and timeless nature of mathematical concepts. The history of mathematics provides evidence of the contrary. Mathematical concepts—even the most elementary ones, like the concept of number—continuously change over time. The objects signified by the ancient Greek concept of arithmos differ from that of number by Renaissance mathematicians, which in turn differ from our current view. One could object that—not mathematics itself—our understanding of mathematical reality changes. Jacob Klein’s landmark study (1934-6), however, focuses precisely on the ontological shift in the number concept. In Greek arithmetic one was not a number, but later it was. After that, the root of two was accepted as a number, and by the end of the sixteenth century, the root of $\sqrt{-15}$ became a number.

4. LOOKING BEHIND THE BARRIER OF SYMBOLIC THINKING

Dealing with the development of symbolic algebra, we must define some terms more explicitly. Let us call algebra an analytical problem-solving method for arithmetical problems in which an unknown quantity is represented by an abstract entity. There are two crucial conditions in this definition: analytical, meaning that the problem is solved by considering some unknown magnitudes hypothetical and deductively deriving statements so that these unknowns can be expressed as a value, and an abstract entity that is used to represent the unknowns. This entity can be a symbol, a figure,
or even a color as, we shall see below. More strictly, symbolic algebra is an analytical problem-solving method for arithmetical and geometrical problems consisting of systematic manipulation of a symbolic representation of the problem. Symbolic algebra thus starts from a symbolic representation of a problem, meaning something more than a short-hand notation. There is no room here to expand on this important difference. Instead, we will focus on one important misunderstanding: “as arithmetical problems are solved algebraically for over 3,000 years, an algebraic equation is a very old concept.” This is not the case, as we shall argue. The symbolic equation is an invention of the sixteenth century.

We are all educated in the symbolic mode of thinking, which is so predominant that it becomes very difficult to grasp how non-symbolic algebra really works. In fact, in the history of mathematics there are many cases in which one completely ignored the difference. Let us take one example of Babylonian algebra. That Babylonians had an advanced knowledge of algebra is a fact that became known rather late—around 1930. Many thousands of clay cuneiform tablets were found that contained either tables with numbers or the solutions to numerical problems. One such tablet is YBC 6967 from Yale University, written in the Akkadian dialect around 1500 B.C. The most prominent scholar who studied and edited these mathematical tablets was Otto Neugebauer (1935-7, 1945). For the problem on YBC 6967, Neugebauer writes the following:

The problem treated here belongs to a well-known class of quadratic equations characterized by the terms igi and igi-bi (in Akkadian igūm and igibūm respectively) (...) We must here assume the product

\[ xy = 60 \] (0.1)

as the first condition to which the unknowns \( x \) and \( y \) are subject. The second condition is explicitly given as

\[ x - y = 7 \] (0.2)

From these two equations, it follows that \( x \) and \( y \) can be found from

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6 Mahony (1980) is one of the few to clarify the distinction. See also my forthcoming “Sixteenth century algebra as a shift in predominant models.”

7 Neugebauer and Sachs (1945, 129-30). The Babylonians used the sexadecimal number system, in which a unit is represented by Neugebauer as 1,0. I have changed this to decimal numbers and added the reconstructed text fragments for easier reading, which leaves the problem text otherwise intact.
a formula which is followed exactly by the text, leading to \( x = 12 \) and \( y = 5 \).

Important here is that Neugebauer claims that equations are explicitly given and that the problem is “found from a formula which is followed exactly by the text.” There are not so many people around who can go back to the cuneiform text and are able to check this claim. Fortunately, Neugebauer added an English translation which allows us to perform the task.

For the explicitly given equation, we read “The \( \text{igibūm} \) exceeds the \( \text{igūm} \) by 7.” This indeed corresponds with the equation (0.3). For the formula, we read “As for you – halve 7, by which the \( \text{igibūm} \) exceeded the \( \text{igūm} \), and the result is 3.5. Multiply together 3.5 with 3.5 and the result is 12.25. To the 12.25, that resulted you add 60, and the result is 72.25. What is the square root of 72.25?—8.5. Lay down 8.5, its equal, and then subtract 3.5 (the \( \text{takīlum} \)), from the one and add it to the other. One is 12, the other 5 (12 is the \( \text{igibūm} \), 5 the \( \text{igūm} \)).” Again, the text seems to correspond with the formula. There are two minor details here: the lay down part sounds a little strange in this context, and Neugebauer adds “we have refrained from translating \( \text{takīlum} \),” because no sense could be given to it.

Recently, Jens Høyrup (2002) published a book that completely overthrows the standard interpretation of Babylonian mathematics and adds a new one. For Høyrup, Babylonian algebra works with geometric figures. This went by completely unnoticed because no figures appear on the tablets. Høyrup’s study, however, is very convincing and its importance for the history of mathematics cannot be overestimated. In this problem, the unknowns \( \text{igibūm} \) and \( \text{igūm} \), are represented by the sides of a rectangle (Høyrup 2002, 55-6). The term \text{product} used by Neugebauer should be read as \text{surface}, \text{square root} as \text{equal side} or the side of a square surface and adding means appending in length. According to Høyrup, the term \( \text{takīlum} \) should be read as \text{make-hold}, or making the sides of a rectangle hold each other. Only within a geometrical interpretation, does it make sense to lay down something. Using a rectangle with sides \( \text{igibūm} \) and \( \text{igūm} \), everything fits together. The \( \text{igibūm} \) is 7 longer than the \( \text{igūm} \). Cutting that part in half leads us to Figure 1.
Figure 1: an example of the geometric algebra from the Babylonians.

If we paste one of the halves below the rectangle at the length of the \( igūm \), we get a figure with the same surface equal to 60.

Figure 2: Cut and paste method for solving quadratic problems.

The part in the lower left corner must be a square, as its sides are both 3 ½. We can thus determine its surface as 12 ¼. The complete figure must also be a square with sides equal to \( igūm \) plus 3 ½. We know that the total surface is 72 ¼—the equal side of that square, therefore, is 8 ½. That leads us to a value of the \( igūm \) being 5. Pasting the cut-out half back to its original place gives a length of the \( igibūm \) of 12.

We are presented here with an interpretation completely different from that of Neugebauer. Høyrup accounts for anomalies in the standard interpretation and gives strong arguments for the reading of terms and actions in the geometrical sense. In this new interpretation, it makes no sense to speak about equations. Babylonian algebra does not solve equations, as the concept of an equation was absent. It fits in with our definition of algebra, however—the method is unquestionably analytical. It uses the unknowns \( igūm \) and \( igibūm \) and they are represented as abstract entities, namely the sides of a rectangle. We cannot blame Neugebauer for his symbolic reading of Babylonian algebra in 1945. Looking behind the barrier
of symbolic thinking proves to be a difficult task. His book was a major contribution to the early history of mathematics, but the history of mathematics has changed in the past decades and conceptual analyses such as Høyrup’s have become the new methodological standard.

5. **DIOPHANTUS: ALGEBRA OR THEORY OF NUMBERS?**

The *Arithmetica* of Diophantus is often considered to be a primary source of European algebra.\(^8\) This interpretation is questionable. The discovery of Diophantus in the fifteenth century had an important influence on the development of symbolic algebra. Its influence, however, is not as decisive as some want us to believe. The prime source for the myth that algebra was invented by Diophantus is Regiomontanus in his Padua lecture of 1464. Just having discovered the manuscript, Regiomontanus describes the *Arithmetica* enthusiastically as a book “in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name. Here and there, the Latins have come in contact with this beautiful art\(^9\)”\(^9\). Later, humanist mathematicians of the sixteenth century, such as Petrus Ramus, were more explicit with the idea that algebra originated with Diophantus and the Arabs learned the art from him.\(^10\) Paradoxically, sixteenth-century humanists continued the program of reassessing mathematics from ancient sources, initiated by the Arabs, and in doing so precisely denied the contribution of the Arabs. Høyrup (1998) traces this evolution over several authors in Renaissance Europe. After Ramus, Bombelli, and Viète were also well acquainted with the *Arithmetica* and carefully avoided references to Arab influences. On the other hand, the Arab roots of algebra have mostly been acknowledged by the Italian abacus tradition from Fibonacci (1202, Boncompagni 1857) through the fifteenth century up to Cardano (1545, Witmer 1968) and the German *cossist* tradition, with Stifel (1544) as a most important author. It is probably thanks to them that we still use the name *algebra* today.

To assess the *Arithmetica*, it is important to draw a distinction between the context of the original text and its adaptations since its discovery by Regiomontanus. The treatment of problems from the *Arithmetica* by Bombelli (1572) and Simon Stevin (1585) are without doubt algebraic. Several editions of the *Arithmetica* have given an algebraic formulation to

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10 Ramus gives a short history of mathematics in his *Scholae mathematicae* (1569).
problems, as has been done with Euclid’s *Elements*. Such reformulation has been historically important for diophantine analysis, but was not necessarily a correct interpretation of the original work. Let us look at Problem 16 from the first book as an illustration. This is a rather simple problem looking for three numbers given their sum two by two:

Table 1: Two Interpretations of Problem 16 from Book 1 of the *Arithmetica* by Diophantus

<table>
<thead>
<tr>
<th>Tannery (1893, 39)</th>
<th>Ver Eecke (1926, 21, my translation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invenire tres numeros tales ut bini simul additi faciant propositos numeros. Oportet propositorum trium dimidiam summam maiorem esse unoquoque horum.</td>
<td>To find three numbers which, taken two by two, form the proposed numbers. It is necessary, however, that half of the sum of the proposed numbers is larger than each one of these numbers.</td>
</tr>
<tr>
<td>Proponatur iam $X_1 + X_2 = 20$, $X_2 + X_3 = 30$, $X_3 + X_1 = 40$</td>
<td>Let us propose that the first number, increased with the second, forms 20 units; that the second, increased with the third, forms 30 units, and that the third, increased with the first, forms 40 units.</td>
</tr>
<tr>
<td>Ponatur $X_1 + X_2 + X_3 = x$</td>
<td>Let us pose that the sum of the three numbers is 1 <em>arithm</em>.</td>
</tr>
<tr>
<td>Quoniam $X_1 + X_2 = 20$, si a $x$ aufero 20, habebo $X_3 = x - 20$</td>
<td>Consequently, because the first number with the second forms 20 units, if we take off 20 units from 1 <em>arithm</em>, we will have the third number: 1 <em>arithm</em> less 20 units.</td>
</tr>
<tr>
<td>Eadem ratione erit $X_1 = x - 30$, $X_2 = x - 40$</td>
<td>For the same reason, the first number will be 1 <em>arithm</em> less 30 units, and the second number will be 1 <em>arithm</em> less 40 units.</td>
</tr>
<tr>
<td>Linquitur summam trium aequari $x$, sed est haec summa $3x - 90$; ista aequentur $x$; fit $x = 45$.</td>
<td>It is necessary, still, that the sum of the three numbers becomes equal to 1 <em>arithm</em>. The sum of the three numbers, however, forms 3 <em>arithms</em> less 90 units. Let us equalize to 1 <em>arithm</em>, and the <em>arithm</em> becomes 45 units.</td>
</tr>
<tr>
<td>Ad positiones. Erit $X_1 = 15$, $X_2 = 5$, $X_3 = 25$. Probatio evidens est.</td>
<td>Let us return to what we posed: the first number will be 15 units, the second will be 5 units, the third will be 25 units, and the proof is clear.</td>
</tr>
</tbody>
</table>
Paul Tannery’s respected critical edition of 1893 gives the original Greek text, reconstructed from several manuscripts, together with a Latin translation. As shown, the Latin translation presents the problem as one of three linear equations with three unknowns \( X_1, X_2, \) and \( X_3 \), and the use of an auxiliary \( x \). The idea of linear equations with several unknowns, however, did not emerge before the mid-sixteenth century. Ver Eecke (1926) performed his French translation from the same Greek text as Tannery, but gives a more cautious interpretation. He does not use any symbols, and draws a distinction between number and \( \text{arithmos} \). The unknowns \( X_1, X_2, \) and \( X_3 \) of Tannery are numbers in the French translation. Instead, the \( \text{arithmos} \) designates the unknown. After stating the problem, Diophantus reformulates the problem expressing the numbers in terms of a chosen unknown.

The interpretation of the \textit{Arithmetica} as symbolic algebra is highly problematic. Even its designation as algebra cannot go without careful qualification. Nesselmann (1842) called it \textit{syncopated algebra} as an intermediate stage between rhetoric and symbolic algebra. This would consist of short-hand notations that had not yet developed to full symbolism. The Greek text uses the letters \( \Delta^r \) and \( K^r \), which have been interpreted by many as the powers of an unknown, \( x^2 \) and \( x^3 \). Ver Eecke simply translates this as \textit{square} and \textit{cube} respectively. And this is without doubt closer to the original context than Tannery’s Latin translation. Diophantus is primarily interested in the properties of numbers. A typical problem sounds like “Find two numbers with their sum and the difference of their squares given” (Book I, Problem 29; Tannery 1893, 65). The aim is to find numbers that satisfy the given property rather than solving the equations \( x + y = 20, x^2 - y^2 = 80 \). All problems of the \textit{Arithmetica} are stated in the general way. A reading of the \textit{Arithmetica} as a general theory of numbers is further emphasized by the character of diophantine problems having an infinity of numbers satisfying a given property. Diophantus’s \textit{Arithmetica} can be equally, or better, understood as a study on the properties of natural numbers than as early algebra. To read the text as an early form of symbolic algebra cannot be reconciled with the definition we have given above.

6. THE COLORFUL ALGEBRA OF THE HINDUS

Hindu tradition has passed down to us several important works on arithmetic and algebra, the importance of which to the development of algebra is still underestimated. The major handicap in drawing a line of influence of Indian sources on the development of Renaissance arithmetic and algebra is its
indirect character and lack of written evidence. We can trace some important paths of transmission for arithmetic and the Hindu-Arabic number system we currently use today. Some Arab texts that were translated into Latin clearly refer to Hindu sources. Early arithmetic books are structurally very similar, for example, to Bhramagupta’s *Brāhma-sphutasiddhānta (BSS)* of 628 AD (Colebrook 1817). There is no known textual evidence, however, that shows a direct influence of Hindu algebra in the West. Comparing many problems treated in Hindu sources as well as in Renaissance algebra, we cannot avoid the particular similarity of both the formulation of the problems and most of the solution methods. Many linear problems solved algebraically in the abacus tradition have their counterpart in Hindu sources, while they are only rarely treated in Arab texts. We can discern an important influence of the oral tradition of recreational problems. Practical and recreational problems have functioned as vehicles for problem prototypes with typical solution patterns. The solution method for typical problems are given as rules in Hindu texts. These rules are mostly formulated in Sanskrit verse, as stanzas or *sūtras*. Given the scarcity and cost of writing aids, memorizing aids in the form of verse was very important in mathematical texts before the age of printing. As an example, consider the following rule for solving linear problems given both in the *BSS* and the *Bīja-Ganita (BG)* of Bhāskarācārya of c. 1150:

Table 2: Solving Linear Problems *BSS* and *Bīja-Ganita (BG)* of Bhāskarācārya

<table>
<thead>
<tr>
<th>Colebrook (1817, 227)</th>
<th>Dvivedi (1902)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtract the first colour [or letter] from the other side of the equation; and the rest of the colours [or letters] as well as the known quantities, from the first side: the other side being then divided by the [coefficient of the] first, a value of the first colour will be obtained. If there be several values of one colour, making in such case equations of them and dropping the denominator, the values of the rest of the colours are to be found from them.</td>
<td>Removing the other unknowns from [the side of] the first unknown and dividing the coefficient of the first unknown, the value of the first unknown [is obtained]. In case of more [values of the first unknown], two and two [of them] should be considered after reducing them to common denominators.</td>
</tr>
</tbody>
</table>

11 Dixit Algorizmi c. 825. For a French translation, see Allard, 1992, 1-22.
12 I have argued this more extensively in “How algebra spoiled Renaissance’s practical and recreational problems” (forthcoming).
In the English rendition of the Sanskrit verses, Colebrook uses the term *equation* but he is not followed by Dvivedi. Instead, Dvivedi uses the terms *unknown* and *coefficient*, which in turn are not used by Colebrook. We can, therefore, cast some doubt about the use of these modern terms. Furthermore, Datta and Singh (1962, II, 9) claim that “in Hindu algebra there is no systematic use of any special term for the coefficient.”

Prthūdakasvāmī (860), Śrīpati (1039) and later Bhāskara (1150), solved linear problems by the use of several colors representing the unknowns. In other cases, flavors such as sweet (*madhura*) or flowers were also used for the same purpose. Solutions were mostly based on rules for prototypical cases, such as the rule of concurrence (*sankramana*) \( \{ x + y = a, x - y = b \} \), or the pulverizer (*Kuttaka*) \( ax - by = c \). In several texts, starting with the BSS, we find reference to *samīcarana*, *samīcarā*, or *samīcriyā* often translated as *equation*. The rationale for this is that sama means *equal* and cri stands for *to do*. As with the terms *aequatio* and *aequationis* in early Latin works on algebra, we should be careful interpreting these terms in the modern way. They basically mean *the act of making even*—an essential operation in the algebraic solution of problems. They do not necessarily mean an equation in the sense of symbolic algebra. The basis of Hindu algebra is to reduce problems to the form of given precepts that provide a proven solution to the problem. The method is algebraic, as it uses abstract entities for the unknowns and is analytical in its approach. The Hindu methods for solving linear problems were transmitted to the West by prototypical problems, mostly of the recreational type, which served as vehicles for the corresponding problem-solving recipes. An example is the case \( \{ x + a = c(y - a), y + b = d(x - b) \} \), which we find in the *Ganitasārasangraha* of Mahāvīra and the *BG*, but also in several fifteenth-century arithmetics under the name *regula augmentationis*.

### 7. **ARAB ALGEBRA**

Arab algebra was introduced in Europe by the translations of the *Algebra* of Mohammed ibn Mūsa al-Kwārizmī by Guglielmo de Lunis, Gerhard von Cremona (1145), and Robert of Chester (1450; Hughes 1981). Most importantly however was the *Liber Abaci* of Fibonacci (1202). Fibonacci devoted the last part of his book to algebra and used mostly problems and solution methods from al-Kwārizmī and Al-Karkhī. Although Arab algebra developed to a high degree of sophistication during the next centuries, it was mostly the content of these early works that were known in Europe. Recent studies have provided us with a new picture on the continuous development
of algebra in the Italian abacus schools between Fibonacci and Luca Pacioli’s *Summa de arithmetica geometria proportioni* (1494) (Franci and Rigatelli 1985). It took about four centuries before the transition to symbolic algebra was completed.

al-Kwārizmī gives solutions to algebraic problems by applying proven procedures in an algorithmic way. The validity of the solution is further demonstrated by geometrical diagrams. In contrast with Babylonian algebra, the method is not geometrical in nature—only the demonstration and interpretation is. As an example let us look at the way al-Kwārizmī solves Case 4 of the quadratic problem that can be represented by the well-known equation, \( x^2 + 10x = 39 \) (Rosen 1831, italics are mine):

For instance, one square, and ten roots of the same, amount to thirty-nine dirhems. That is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: *you halve the number of the roots*, which in the present instance yields five. *This you multiply by itself;* the product is twenty-five. *Add this to thirty-nine;* the sum is sixty-four. *Now take the root of this,* which is eight, and *subtract from it half the number of the roots,* which is five; the remainder is three. *This is the root of the square which you sought for;* the square itself is nine.

If we write the case as \( x^2 + bx = c \), the solution fully depends on the application of the procedure that corresponds to \( \sqrt{\left( \frac{b}{2} \right)^2 + c - \frac{b}{2}} \). Solving problems in Arab algebra consists of formulating the problem in terms of the unknown and reducing the form to a known case. Methods for solving quadratic problems were given before in Babylonian and Hindu algebra. Again, we see no equations in Arab algebra. The explicit treatment of operations on polynomials is new, however. The basic operations of addition, subtraction, multiplication, and division, which were applied before to numbers, are now extended to an aggregation of algebraic terms. A further expansion of these operations would lead to the concept of a symbolic equation in the sixteenth century.

8. **THE EMERGENCE OF THE CONCEPT OF AN EQUATION**

So, what is it that constitutes the concept of an equation? I propose to adopt an operational definition of the term to reconstruct the historical emergence of the concept. We now consider an equation as a mathematical object on which certain operations are allowed. Let us, therefore, look at the precise
point in time in which an equation is named, consistently used, and operated upon as a mathematical object. As said before, the use of the term *aequatio* is not a sufficient condition for the existence of an equation. The observation that two polynomials are numerically identical does not in itself constitute an equation. An operation on an equation, however, would be. The first historical instance that I could find is in Cardano’s *Practica arithmetice* (1539, f. 91r).

This is probably the most important page in the development of symbolic algebra, as it combines two important conceptual innovations in a single problem solution—the use of a second unknown and the first operation on an equation.

Figure 3: The first operation on an equation in Cardano’s *Practica arithmetice* of 1539.

Cardano uses *co.* for the primary unknown and *quan.* for a secondary one. We can justly write this as $x$ and $y$ without misinterpreting the original context. In the example given in Figure 3, Cardano manipulates several polynomials, but at some point moves to equations.

We find 7 *co.* aequales 151 p. 27 *quan.* $(7x = 151 + 27y)$ and 10 *co.* aequales 1018 p. 18 *quan.* $(10x = 1018 + 18y)$. He divides these equations by 7 and 10 respectively, without explicitly saying so. By equating both, however, he arrives at $80 \frac{8}{35} = 2 \frac{2}{35} y$, which he explicitly multiplies by 35 to arrive at $72y = 2808$ or $y = 39$ (misprinted as $2008 = 72y$). From this moment on, algebra changed drastically. Cardano’s book was widely read and several authors built further on this milestone. Stifel (1545) introduced the letters 1A, 1B, and 1C to differentiate multiple unknowns, which removed most of the ambiguities from earlier notations. It was Johannes Buteo (1559), however, who
established a method for solving simultaneous linear equations by systematically substituting, multiplying, and subtracting equations to eliminate unknowns. These developments between 1539 and 1559 constituted the concept of a symbolic equation. The equation became, not only a representation of an arithmetical equivalence, but also represented the combinatorial operations possible on the symbolic structure. This paved the road for Viète (1591) and Harriot to study the structure of symbolic equations.

9. CONCLUSION

We treated 3000 years of algebra in a few paragraphs with the risk of over simplification. One important conclusion emerges, however, —at some point in history there was a dramatic change in the way arithmetical problems were solved. By the second half of the sixteenth century, algebraic problem solving became the systematic manipulation of symbolic equations. We argue that the concept of an equation, as we understand it today, did not exist before that time. The development of sixteenth-century algebra is one of those occasions in which we see the birth of a new important concept in mathematics. Algebra did exist before, but functioned in a different way. Symbolic algebra, as it is currently taught in secondary education, is only one aspect of algebraic practice. While symbolic algebra may be the most efficient kind in problem solving, it is not always the most adequate one for teaching basic algebraic concepts to children. Luis Radford (1995, 1996, 1997) has demonstrated how procedures from the pre-symbolic abacus tradition can contribute to a better didactic understanding of the use of multiple unknowns. Joëlle Vlassis (2002) points out that conceptual difficulties with negative numbers originate from symbolic algebra and argues for a non-symbolic way to convey the concept. Our current conceptualization of algebra may confuse students in their first exposure to symbolic problem solving. An approach to improving such a situation, which has found some recognition during the past years, is to employ a plurality of methods. A new concept, method, or theorem, explained in multiple ways, is more likely to reach a broader range of students. Some students have difficulties with purely symbolic accounts of mathematics. Others are weak in spatial representations. Still others need numerical examples to be able to grasp abstract relations and functions. Teaching concepts by a plurality of methods levels out these difficulties. The history of mathematics provides a vast repository of alternative cases, representations, and methods.

An additional consequence of a plurality of methods and conceptualizations is situated on the philosophical level. If one thing should
be clear from our 3,000-year overview of algebra, it is that mathematics is subject to a historical process, that is based not only on the insights of some inventive individuals but also involves socio-cultural aspects of mathematical practice. The predominant view passed on by mathematics education hides implicit values on the superiority of modern ideas over past ones, and possibly of Western concepts over non-Western ones. Again, the history of mathematics shows that mathematics has always adapted to the needs of society. Mathematics was born in the Fertile Crescent, extending to the belt from North Africa to Asia, where wild seeds were large enough and mammals capable of employable domestication. Modern algebra fertilized in the mercantile context of merchants and craftsman in Renaissance Italy. Several important figures in the development of symbolic algebra wrote also on bookkeeping, as well as on algebra often in one and the same volume. If we accept that double-entry bookkeeping emerged in the fifteenth century as a result of the expanding commercial structures of sedentary merchant in Renaissance Italy, why not consider symbolic algebra within the same context? Ideas should be interpreted within the historical context in which they emerged and perhaps their superiority is dependent on the degree in which they adapted to the needs of society.

The idea of an objective reality of mathematical concepts, therefore, evades the veracity of conceptual dynamics and conceptual problems in mathematics. Conceptual dynamics challenges mathematical realism with some awkward questions. Take the simple concept of an algebraic equation as we have explored it. Of the different historical meanings of an equation, which one corresponds with the metaphysical object separate from human mathematical practice and understanding? If there have been different meanings for a given concept, such as an algebraical unknown, do they all correspond with the object of an unknown for a realist, or only our current conceptualization? If so, what is the ontological status of historical conceptualizations? What about inconsistent conceptualizations? Time and again, there have been serious crises in the conceptual foundations of mathematics. There have been inconsistent theories, such as the early use of analysis and set theory, which have existed for several decades. It is precisely in times of crisis and conceptual difficulties that new ideas emerge.

13 For an eye-opening study on the relationship between these coincidental factors and the development of culture and thus mathematics, see the excellent work of Jared Diamond, 1996.
14 Between 1494 and 1586: Luca Pacioli, Grammateus, Valentin Mennher, Elcius Mellema, Nicolas Petri and Simon Stevin.
15 An important case study on crisis in mathematics is Carl Boyer, The History of the Calculus and Its Conceptual Development, 1959. As the title suggests, Boyer concentrates on the conceptual difficulties in developing the modern ideas of calculus.
and breakthroughs are made. According to Lakatos (1976, 140) such periods are “the most exciting from the historical point of view and should be the most important from the teaching point of view.”

The aim of this paper is to show that the history of mathematics offers ample opportunities to illustrate the plurality of methods and the dynamics of concepts in mathematics. Integrating threads of conceptual development of mathematics in classroom teaching contributes to students’ philosophical attentiveness. Such examples will alert students of the relativity of mathematical methods, truth, and knowledge and will put mathematics back in the perspective of time, culture, and context. The conceptual history of mathematics provides excellent opportunities to convey the basic epistemological and ontological questions of philosophy of mathematics in mathematics education.

REFERENCES


